

3D Formulas for Spreadsheet

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Abstract:

I will in this paper present a new model for a three dimensional coordinate system in which it is possible to visualize three dimensional objects. In order to find the two dimensional coordinates needed, I use the principle of parallel perspective for the unit vectors combined with the principle of the central perspective and a formula for the vanishing points. The two formulas x and y for a general point (x_1, y_1, z_1) is found by the use of linear algebra. The formulas are such that they can be used in any spreadsheet and visualized by connected scatter plots. Any object visualized this way can be rotated by two independent variables, horizontal angle h and vertical angle v .

Introduction

The objective of this article is to construct a three dimensional coordinate system and to present formulas for the x - and y -coordinate of a general 3d point $X(x_1, y_1, z_1)$ which will be shown in the coordinate system. By the use of connected scatter plots, it will be possible to visualize perspective three dimensional objects for which the coordinates are given. These formulas can be used in any spreadsheet, such as Excel or dynamic geometry software packages such as Ti-Nspire. The drawings can be rotated and studied from any point of view subject to only two independent variables, horizontal angle h ranging from 0 to 2π and vertical angle v ranging from $-\pi$ to π . Figure 5 contains screenshots from an object drawn and rotated in Excel.

In Dahan, J. [4] and [5], the principle of the Central Perspective according to Guidobaldo's theorem is thoroughly studied. Parallel lines do not appear parallel in perspective geometry, in fact they intersect in vanishing points. I will first use the parallel principle to find the three unit vectors $\mathbf{i}=\text{OI}$, $\mathbf{j}=\text{OJ}$ and $\mathbf{k}=\text{OK}$ and then use the principle of the central perspective to find the three corresponding vectors to the vanishing points OP , OQ and OR . These six points together with linear algebra, is all that is needed to find the coordinates for a general point X . My approach differs from the traditional one in the sense the coordinates of the point X is not constructed, the principle of intersecting parallel lines of the central perspective is used by finding equations that I solve. In Figure 1 it is possible to see the six points I will first find the coordinates for. When these have been explained I will start the chase for the point X .

There is a one to one connection between the coordinates of a point $A(a_1, a_2)$ and the vector from the origin to the point, $\text{OA}[a_1, a_2]$. Finding the vector is therefore also finding the point. A vector from a point $A(a_1, a_2)$ to a point $B(b_1, b_2)$ will be named AB with coordinates $[b_1-a_1, b_2-a_2]$.

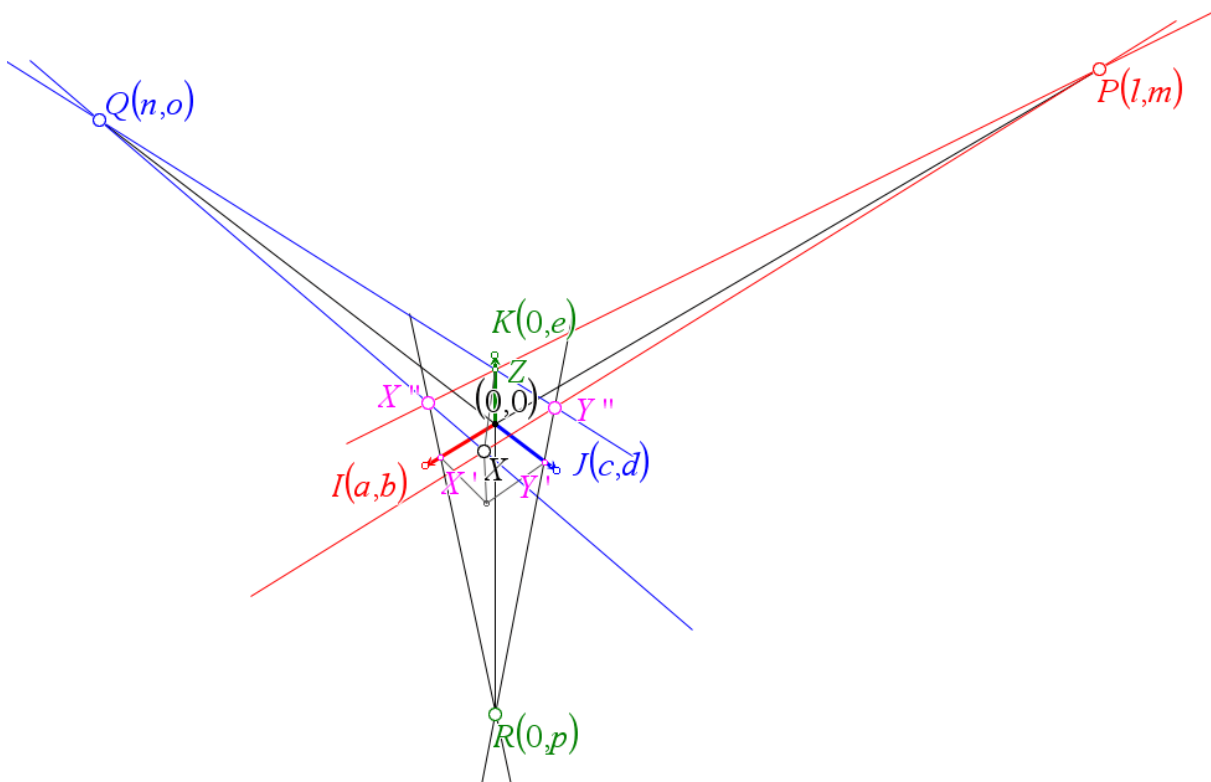


Figure 1; the unit vectors and the vanishing points together with the point X we want to find

Step 1.

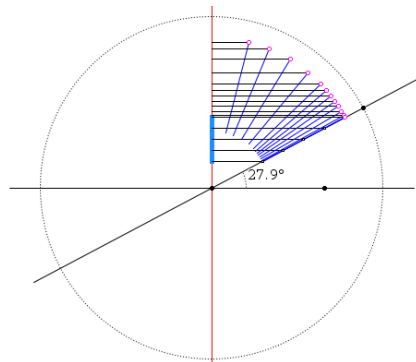


Figure 2: The principle used in vertical rotation

The formulas for the unit vectors are very straightforward. The coordinates for \mathbf{i} are $[\sin(\mathbf{h}), \cos(\mathbf{h}) \cdot \sin(\mathbf{v})]$. If I omit the vertical rotation a point $(\sin(\mathbf{h}), \cos(\mathbf{h}))$ will construct a circle as \mathbf{h} moves from $-\pi$ to π . In order to understand vertical rotation, consider what happens in Figure 2.

The red vertical line represents the screen, my eye is to the right of the figure and when an object like the blue segment is rotated towards me, what is seen on screen is the image of the segment shown in light blue. This is where $\sin(\mathbf{v})$ comes into the formula. In this part of the construction I use the parallel principle. I have illustrated this in Figure 3 by calculating where \mathbf{i} -vector is in red. I

have also included the locus of the endpoint for **i**-vector as we change the angle **h**. This locus is an ellipse which will turn into a circle (see light blue circle in figure 3) for **v** equal to $-\pi/2$ and $\pi/2$. As for the unit vector **j**, the coordinates for this vector is found by just adding $\pi/2$ to the angle **h**. Finally the unit vector **k** will be on the y-axis with coordinate $\cos(v)$.

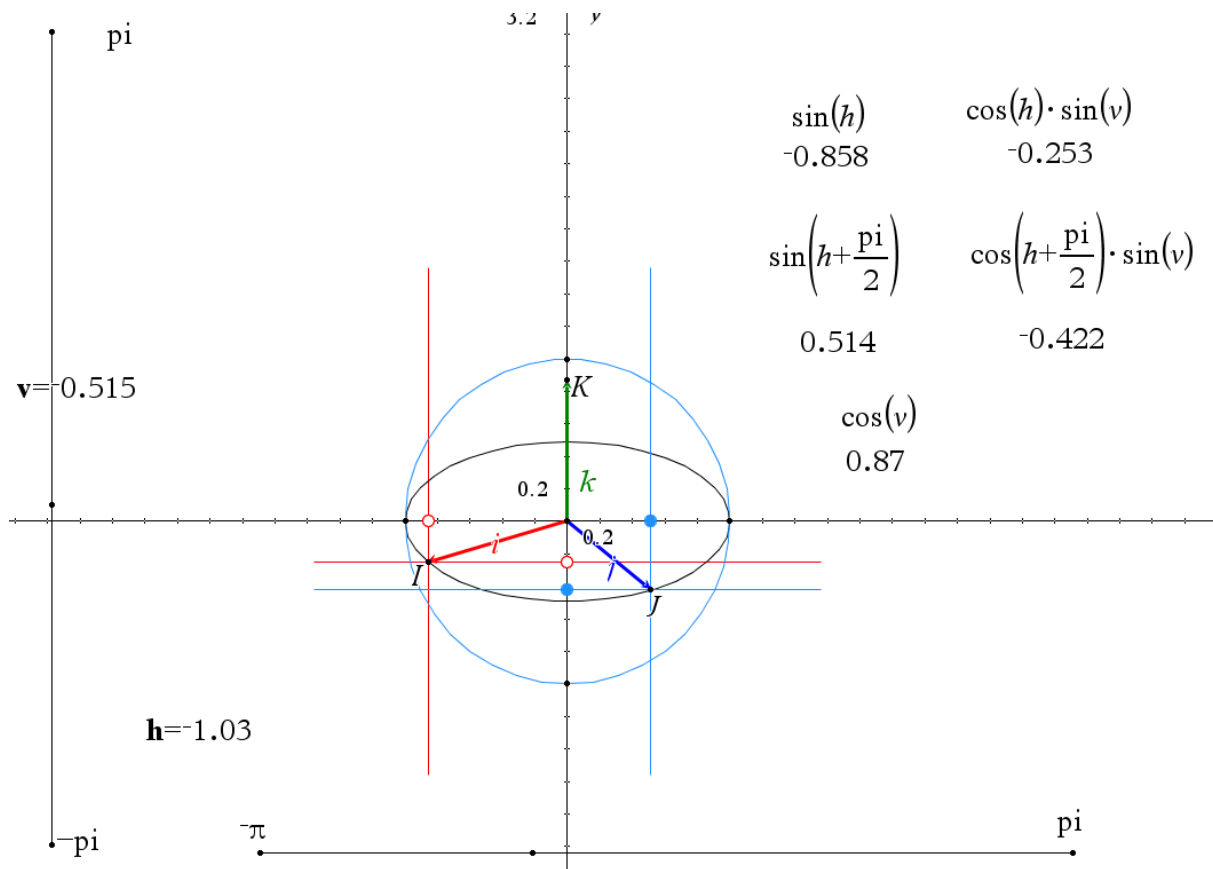


Figure 3: Formulas for the unit vectors

Step 2.

In order to construct the three vanishing points P, Q and R, the following criteria must be obeyed. The vectors OP, OQ and OR must be parallel to the unit vectors, the magnitudes of the vectors should approach ∞ as the unit vectors approach 1. They should always point away from the viewer as **h** and **v** are changed (see appendix for Boolean formulas). The coordinates for OP will be

$$OP = \mathbf{i} \cdot t = [\sin(h), \cos(h) \cdot \sin(v)] \cdot \frac{w}{\sqrt{\sin^2(h) + \cos^2(h) \cdot \sin^2(h)} - 1}$$

The number $t = \frac{w}{\sqrt{\sin^2(h) + \cos^2(h) \cdot \sin^2(h)} - 1}$ with w a constant that moves the vanishing point farther away if I want, contains the magnitude of **i** minus 1 in the denominator. That will ensure that OP will tend to ∞ as the magnitude of **i** tend to 1. OQ will similarly be multiplied with the same number and hence be $OQ = \mathbf{j} \cdot t = \left[\sin\left(h + \frac{\pi}{2}\right), \cos\left(h + \frac{\pi}{2}\right) \cdot \sin(v) \right] \cdot \frac{w}{\sqrt{\sin^2(h) + \cos^2(h) \cdot \sin^2(h)} - 1}$ and $OR = \mathbf{k} \cdot s = [0, \cos(v)] \cdot \frac{w}{|\cos(v)| - 1}$.

OP and OQ will reach their maxima with a displacement of $\pi/2$.

It is worth noting that the number w must be big enough to ensure that the vanishing points are farther away than the coordinates of the object we want to study. If w is too small, the idea of a distant vanishing point will not be respected and the object I intend to visualize will be distorted. In Dahan, J. [4], it is stated that a consequence of the Central Perspective is that the vanishing points P and Q move on a straight line as h is changed. Due to the fact that I decide where the vanishing points should be by deciding the size of w , P and Q will not be on a straight line, but they will follow the same path, as can be seen in Figure 4. Due to this fact, my construction is not the Central perspective principle, but rather a mixture of the parallel and the central perspective which makes it a very close approximation.

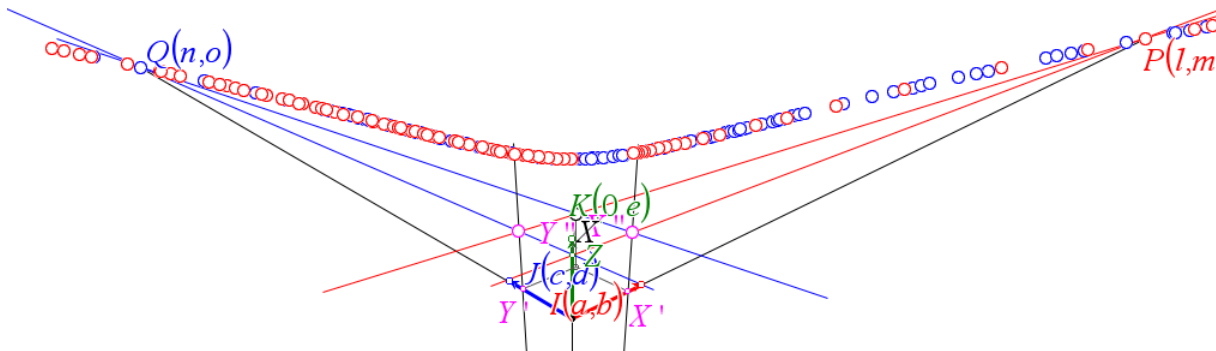


Figure 4; the path of the vanishing points P and Q for a small value of w .

I now have six vectors which will be all I need to use linear algebra to find the coordinates for a general three dimensional point (x_1, y_1, z_1) ;

$$i = [\sin(h), \cos(h) \cdot \sin(v)]$$

$$j = \left[\sin\left(h + \frac{\pi}{2}\right), \cos\left(h + \frac{\pi}{2}\right) \cdot \sin(v) \right]$$

$$k = [0, \cos(v)]$$

$$OP = \left[\frac{\sin(h) \cdot w}{\sqrt{\sin^2(h) + \cos^2(h) \cdot \sin^2(h)} - 1}, \frac{\cos(h) \cdot \sin(v) \cdot w}{\sqrt{\sin^2(h) + \cos^2(h) \cdot \sin^2(h)} - 1} \right]$$

$$OQ = \left[\frac{\sin\left(h + \frac{\pi}{2}\right) \cdot w}{\sqrt{\sin^2(h) + \cos^2(h) \cdot \sin^2(h)} - 1}, \frac{\cos\left(h + \frac{\pi}{2}\right) \cdot \sin(v) \cdot w}{\sqrt{\sin^2(h) + \cos^2(h) \cdot \sin^2(h)} - 1} \right]$$

$$OR = \left[0, \frac{\cos(v) \cdot w}{|\cos(v)| - 1} \right]$$

Step 3.

In order for me to find the coordinates for the point X'' (see Figure 1) in my coordinate system, I “walk” to X'' in two ways using the property that two vectors are parallel if and only if one can be

expressed as a number times the other. The coordinates for the points X', Y' and Z in Figure 1 are: X'(x₁a, x₁b), Y'(y₁c, y₁d), Z(0, z₁e) because OI [a, b], OJ [c, d] and OK [0, e] are the unit vectors.

I have $OX'' = OR + t_1 \cdot RX' = [0, p] + t_1 \cdot [x_1 a, x_1 b - p]$ and also $OX'' = OP + t_2 \cdot PZ = [l, m] + t_2 \cdot [-l, z_1 e - m]$. Two vectors are equal if and only if the coordinates are equal and therefore I have the following set of equations to solve.

$$\begin{aligned} t_1 \cdot x_1 \cdot a &= l - t_2 \cdot l \\ p + t_1 \cdot (x_1 \cdot b - p) &= m + t_2 \cdot (z_1 \cdot e - m) \end{aligned}$$

Solving this set with respect to t₁ and t₂ reveals that $t_1 = \frac{(e \cdot z_1 - p) \cdot l}{a \cdot (e \cdot z_1 - m) \cdot x_1 + (b \cdot x_1 - p) \cdot l}$. Substituting this

into the vector OX'' gives me the coordinates for X'' = $\left(\frac{a \cdot e \cdot l \cdot x_1 \cdot z_1 - a \cdot l \cdot p \cdot x_1}{a \cdot e \cdot x_1 \cdot z_1 - a \cdot m \cdot x_1 + b \cdot l \cdot x_1 - l \cdot p}, \frac{a \cdot e \cdot p \cdot x_1 \cdot z_1 - a \cdot m \cdot p \cdot x_1 + b \cdot e \cdot l \cdot x_1 \cdot z_1 - e \cdot l \cdot p \cdot z_1}{a \cdot e \cdot x_1 \cdot z_1 - a \cdot m \cdot x_1 + b \cdot l \cdot x_1 - l \cdot p} \right)$.

Similarly to find Y'' I have $OY'' = OR + s_1 \cdot RY' = [0, p] + s_1 \cdot [y_1 c, y_1 d - p]$ and $OY'' = OQ + s_2 \cdot QZ = [n, o] + s_2 \cdot [-n, z_1 e - o]$. Again I have a set of equations:

$$\begin{aligned} s_1 \cdot y_1 \cdot c &= n - s_2 \cdot n \\ p + s_1 \cdot (y_1 \cdot d - p) &= o + s_2 \cdot (z_1 \cdot e - o) \end{aligned}$$

Now $s_1 = \frac{(e \cdot z_1 - p) \cdot n}{c \cdot (e \cdot z_1 - o) \cdot y_1 + (d \cdot y_1 - p) \cdot n}$ and when I substitute this into vector OY'', I have

$$Y'' = \left(\frac{c \cdot e \cdot n \cdot y_1 \cdot z_1 - c \cdot n \cdot p \cdot y_1}{c \cdot e \cdot y_1 \cdot z_1 - c \cdot o \cdot y_1 + d \cdot n \cdot y_1 - n \cdot p}, \frac{c \cdot e \cdot p \cdot y_1 \cdot z_1 - c \cdot o \cdot p \cdot y_1 + d \cdot e \cdot n \cdot y_1 \cdot z_1 - e \cdot n \cdot p \cdot z_1}{c \cdot e \cdot y_1 \cdot z_1 - c \cdot o \cdot y_1 + d \cdot n \cdot y_1 - n \cdot p} \right)$$

Finding these two points will facilitate my main objective, i.e. to find the coordinates for the point X.

I have that $OX = OP + t_3 \cdot PY'' = [l, m] + t_3 \cdot \left[\frac{c \cdot e \cdot n \cdot y_1 \cdot z_1 - c \cdot n \cdot p \cdot y_1}{c \cdot e \cdot y_1 \cdot z_1 - c \cdot o \cdot y_1 + d \cdot n \cdot y_1 - n \cdot p} - l, \frac{c \cdot e \cdot p \cdot y_1 \cdot z_1 - c \cdot o \cdot p \cdot y_1 + d \cdot e \cdot n \cdot y_1 \cdot z_1 - e \cdot n \cdot p \cdot z_1}{c \cdot e \cdot y_1 \cdot z_1 - c \cdot o \cdot y_1 + d \cdot n \cdot y_1 - n \cdot p} - m \right]$

and $OX = OQ + t_4 \cdot QX'' = [n, o] + t_4 \cdot \left[\frac{a \cdot e \cdot l \cdot x_1 \cdot z_1 - a \cdot l \cdot p \cdot x_1}{a \cdot e \cdot x_1 \cdot z_1 - a \cdot m \cdot x_1 + b \cdot l \cdot x_1 - l \cdot p} - n, \frac{a \cdot e \cdot p \cdot x_1 \cdot z_1 - a \cdot m \cdot p \cdot x_1 + b \cdot e \cdot l \cdot x_1 \cdot z_1 - e \cdot l \cdot p \cdot z_1}{a \cdot e \cdot x_1 \cdot z_1 - a \cdot m \cdot x_1 + b \cdot l \cdot x_1 - l \cdot p} - o \right]$

Solving the system of equations I get from these two expressions for OX in order to find t₃ requires some tough algebra. However, it has been done by the use of Computer Algebra System (CAS) and when substituting the value of t₃ into the first expression for OX the first coordinate of X is

$$\frac{(a(c(l(o-p) + (m-p)n)b_1 - (d \cdot b_1 - p)l \cdot n)a_1 - (b \cdot a_1 - p)c \cdot l \cdot n \cdot b_1)(e \cdot c_1 - p)}{a(c(e(m+o-2p)c_1 - m \cdot o + p^2)b_1 - (d \cdot b_1 - p)(e \cdot c_1 - m)n)a_1 - (b \cdot a_1 - p)(c(e \cdot c_1 - o)b_1 + (d \cdot b_1 - p)n)l}$$

and the second coordinate is

$$\frac{a(c(e(m \cdot o - p^2)c_1 - (m(2o-p) - o \cdot p)p)b_1 - (d \cdot b_1 - p)(e \cdot c_1 - m)n \cdot p)a_1 - (b \cdot a_1 - p)(c(e \cdot c_1 - o)p \cdot b_1 + (d \cdot b_1 - p)e \cdot n \cdot c_1)l}{a(c(e(m+o-2p)c_1 - m \cdot o + p^2)b_1 - (d \cdot b_1 - p)(e \cdot c_1 - m)n)a_1 - (b \cdot a_1 - p)(c(e \cdot c_1 - o)b_1 + (d \cdot b_1 - p)n)l}$$

As was pointed out initially in this paper, the six points I base this construction on is in fact a coordinate system that I can use to enter any three dimensional point in. It is for example possible

to have the coordinates translated by rotation matrices based on rotation around the z-axis, the y-axis and the x-axis before they are evaluated by my formulas and drawn in the coordinate system as Figure 5 shows.

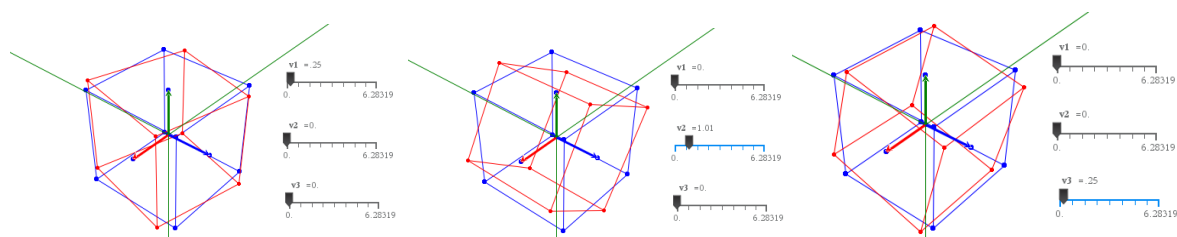


Figure 5: The rotation of a cube using rotation matrices inside my coordinate system.

Conclusion

While many 2d representations of 3d objects are based on geometry, my representation is based on linear algebra and the use of the principles of the Central and the Parallel Perspective. In Banacka, J. [1], [2] and [3], 3d constructions in Excel are presented. He explains thoroughly how the Central Perspective can be used to construct 3d objects in Excel through a traditional use of these principles. The result of his constructions is similar to the ones presented in this paper, but the formulas are different due to the fact that my formulas are based on a mixture of the Central and the Parallel Perspective and an innovative use of algebra to develop the formulas.

The figures in this article have been constructed in TI-Nspire CAS but the formulas are of a general nature and the objects can thus be presented in any spreadsheet, e.g. Excel (see Figure 6). The formulas needed are found in the appendix. The language used for Boolean formulas vary from software to software. In the appendix the language used is that of TI-Nspire and Excel. Be aware of the possibility, that with scatter plots in spreadsheet, it is possible to split up the connected scatter plot by using an empty row between sets of numbers. The formulas in the appendix might seem cumbersome, but they only consist of the coordinates for the three vanishing points and what the angles \mathbf{h} and \mathbf{v} must be in order for the coordinates to change sign and thus making sure the vanishing points are away from the viewer.

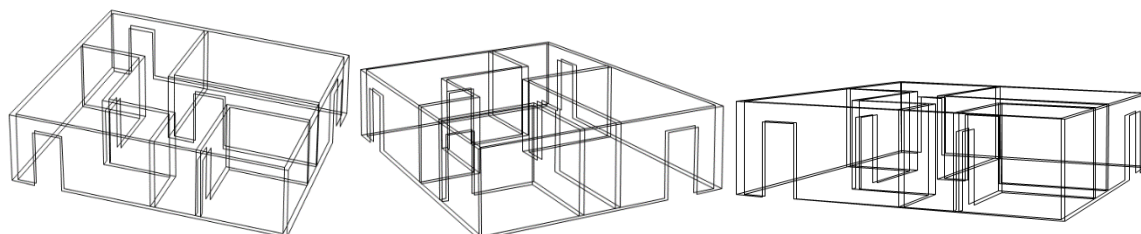


Figure 6: Object created and presented as a scatter plot in Excel.

Appendix

$$l := \text{when} \left(\left(-\pi < v < \frac{-\pi}{2} \text{ or } \frac{\pi}{2} < v < \pi \right) \text{ and } \left(0 < h < \frac{\pi}{2} \text{ or } \frac{3\pi}{2} < h < 2\pi \right) \text{ or } \frac{-\pi}{2} < v < \frac{\pi}{2} \text{ and } \frac{\pi}{2} < h < \frac{3\pi}{2}, \right. \\ \left. \frac{\sin(h) \cdot w}{\sqrt{\sin^2(h) + \cos^2(h) \cdot \sin^2(h)} - 1}, \frac{-\sin(h) \cdot w}{\sqrt{\sin^2(h) + \cos^2(h) \cdot \sin^2(h)} - 1} \right)$$

$$m := \text{when} \left(\left(-\pi < v < \frac{-\pi}{2} \text{ or } \frac{\pi}{2} < v < \pi \right) \text{ and } \left(0 < h < \frac{\pi}{2} \text{ or } \frac{3\pi}{2} < h < 2\pi \right) \text{ or } \frac{-\pi}{2} < v < \frac{\pi}{2} \text{ and } \frac{\pi}{2} < h < \frac{3\pi}{2}, \right. \\ \left. \frac{\cos(h) \cdot \sin(v) \cdot w}{\sqrt{\sin^2(h) + \cos^2(h) \cdot \sin^2(h)} - 1}, \frac{-\cos(h) \cdot \sin(v) \cdot w}{\sqrt{\sin^2(h) + \cos^2(h) \cdot \sin^2(h)} - 1} \right)$$

$$n := \text{when} \left(0 < h < \pi \text{ and } \frac{-\pi}{2} < v < \frac{\pi}{2} \text{ or } \pi < h < 2\pi \text{ and } \left(-\pi < v < \frac{-\pi}{2} \text{ or } \frac{\pi}{2} < v < \pi \right), \right. \\ \left. \frac{-\sin\left(h + \frac{\pi}{2}\right) \cdot w}{\sqrt{\sin^2(h) + \cos^2(h) \cdot \sin^2(h)} - 1}, \frac{\sin\left(h + \frac{\pi}{2}\right) \cdot w}{\sqrt{\sin^2(h) + \cos^2(h) \cdot \sin^2(h)} - 1} \right)$$

$$o := \text{when} \left(0 < h < \pi \text{ and } \frac{-\pi}{2} < v < \frac{\pi}{2} \text{ or } \pi < h < 2\pi \text{ and } \left(-\pi < v < \frac{-\pi}{2} \text{ or } \frac{\pi}{2} < v < \pi \right), \right. \\ \left. \frac{-\cos\left(h + \frac{\pi}{2}\right) \cdot \sin(v) \cdot w}{\sqrt{\sin^2(h) + \cos^2(h) \cdot \sin^2(h)} - 1}, \frac{\cos\left(h + \frac{\pi}{2}\right) \cdot \sin(v) \cdot w}{\sqrt{\sin^2(h) + \cos^2(h) \cdot \sin^2(h)} - 1} \right)$$

$$p := \text{when} \left(-\frac{\pi}{2} < v < 0 \text{ or } \frac{\pi}{2} < v < \pi, -\frac{w \cdot |\cos(v)|}{1 - |\cos(v)|}, \frac{w \cdot |\cos(v)|}{1 - |\cos(v)|} \right)$$

In the Excel file I made to illustrate an object such as the one in Figure 6, the angle **h** is stored in m1, w in m2 and v in m3. The language to enter the variables **l**, **m**, **n**, **o** and **p** in a cell in Excel is this

```
l: =IF(OR(AND(OR(AND(M$3>PI());M$3<PI()/2);AND(M$3>PI()/2;M$3<PI()))OR(AND(M$1>0;M$1<PI()/2);AND(M$1>3*PI()/2;M$1<2*PI())));AND(AND(M$3>-PI()/2;M$3<PI()/2);AND(M$1>PI()/2;M$1<3*PI()/2));(-((SIN(M$1)*M$2)/(((COS(M$1))^2)*(SIN(M$3))^2)+(SIN(M$1))^2)^(0,5)-1));((SIN(M$1)*M$2)/(((COS(M$1))^2)*(SIN(M$3))^2)+(SIN(M$1))^2)^(0,5)-1)))
```

m: =IF(OR(AND(OR(AND(M\$3>-PI();M\$3<-PI()/2);AND(M\$3>PI()/2;M\$3<PI()));OR(AND(M\$1>0;M\$1<PI()/2);AND(M\$1>3*PI()/2;M\$1<2*PI())));AND(AND(M\$3>-PI()/2;M\$3<PI()/2);AND(M\$1>PI()/2;M\$1<3*PI()/2)));(-((COS(M\$1)*SIN(M\$3)*M\$2)/(((COS(M\$1))^2*(SIN(M\$3))^2+(SIN(M\$1))^2)^0,5)-1));((COS(M\$1)*SIN(M\$3)*M\$2)/(((COS(M\$1))^2*(SIN(M\$3))^2+(SIN(M\$1))^2)^0,5)-1)))

n: =IF(OR(AND(AND(M\$1>0;M\$1<PI());AND(M\$3>-PI()/2;M\$3<PI()/2));AND(AND(M\$1>PI();M\$1<2*PI())OR(AND(M\$3>-PI();M\$3<-PI()/2);AND(M\$3>PI()/2;M\$3<PI()))));(-((COS(M\$1)*M\$2)/(((COS(M\$1))^2+(SIN(M\$1))^2*(SIN(M\$3))^2)^0,5)-1));((COS(M\$1)*M\$2)/(((COS(M\$1))^2+(SIN(M\$1))^2*(SIN(M\$3))^2)^0,5)-1)))

o: =IF(OR(AND(AND(M\$1>0;M\$1<PI());AND(M\$3>-PI()/2;M\$3<PI()/2));AND(AND(M\$1>PI();M\$1<2*PI());OR(AND(M\$3>-PI();M\$3<-PI()/2);AND(M\$3>PI()/2;M\$3<PI()))));((SIN(M\$1)*SIN(M\$3)*M\$2)/(((COS(M\$1))^2+(SIN(M\$1))^2*(SIN(M\$3))^2)^0,5)-1));(-((SIN(M\$1)*SIN(M\$3)*M\$2)/(((COS(M\$1))^2+(SIN(M\$1))^2*(SIN(M\$3))^2)^0,5)-1)))

p: =IF(OR(AND(M\$3>-PI()/2;M\$3<0);AND(M\$3>PI()/2;M\$3<PI()));(-((M\$2*ABS(COS(M\$3)))/(1-ABS(COS(M\$3)))));((M\$2*ABS(COS(M\$3)))/(1-ABS(COS(M\$3)))))

When the formulas for the general point X is entered in a cell the variables from **a** to **e** and those above must be replaced by the corresponding cells they are stored in.

I want to underline that there is a difference in settings for Excel as far as I know about the use of commas, full stop sign or semicolon. You are advised to check the settings on your machine.

Acknowledgement

I want to thank Professor Jean-Jacques Dahan for his much appreciated help in my writing this paper. His knowledge of The Central Perspective and dynamic geometry has been a great help and inspiration for me.

Software Packages:

[TI-Nspire] TI-Nspire CAS Teacher Version, version 3.2.0.1219, a product from Texas Instruments

[Microsoft Excel] Microsoft Office Standard 2010, version 14.0.7180.5002

Supplementary Electronic Material:

[Link to the Excel file.](#)

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